Probabilistically implementing nonlocal operation using non-maximally entangled state

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arXiv:quant-ph/0501107v1 20 Jan 2005

We develop the probabilistic implementation of a nonlocal gate $\exp[i\xi\sigma_{n_A}\sigma_{n_B}]$ and $\xi \in [0, \frac{\pi}{4}]$, by using a single non-maximally entangled state. We prove that, nonlocal gates can be implemented with a fidelity greater than 79.3% and a consumption of less than 0.969 ebits and 2 classical bits, when $\xi \leq 0.353$. This provides a higher bound for the feasible operation compared to the former techniques [9, 14, 16]. Besides, gates with $\xi \geq 0.353$ can be implemented with the probability 79.3% and a consumption of 0.969 ebits, which is the same efficiency as the distillation-based protocol [14, 16], while our method saves extra classical resource. Gates with $\xi \to 0$ can be implemented with nearly unit probability and a small entanglement. We also generalize some application to the multiple system, where we find it is possible to implement certain nonlocal gates between many non-entangled partners using a non-maximally multiple entangled state.

Entanglement has been examined as an essential resource in most applications of quantum information such as enhanced classical communication, dense coding and quantum cryptography [1]. Of all tasks above, the implementation of nonlocal quantum operation on spatially distributed systems is a considerable aspect, especially in quantum computation. This is because, all in all, the only performance by a quantum computer is proved to be the collective unitary operation, which is also able to create entanglement between distributed groups. In particular, the latter effect implies that one can realize the above tasks, such as quantum teleportation [2] starting from one entangling operation. Besides the nonlocal Hamiltonians directly results in the dynamical evolution of distributed quantum systems, thus it is also significative to explore other properties such as the structure and interconvertability of nonlocal gates.

In fact, many results have been reported in this direction [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Here, a prime problem is the efficiency for implementing a given nonlocal gate. It is accepted that at least the consumption of 1 ebit and 2 cbits is necessarily required to faithfully implement a general nonlocal operation [8]. On the other hand, a remarkable progress has been made by [9], which says gates with small ξ can be implemented by using a lower entanglement and a more classical consumption. However the scheme therein is inefficient for the gates with larger ξ, and it usually requires an excessive amount of classical resource because of plenty of quantum channels required. This deficiency has been made up by the recent work in [14], who employ a probabilistic protocol to implement the nonlocal gates, with a high fidelity when ξ is small. The required classical communication therein is 2 bits, while the entanglement can be made arbitrarily small since it relates to the success fidelity. However, the best attainable fidelity will lower down rapidly with the increasing ξ, thus it is irresponsible to a majority of useful gates. A higher fidelity is needed for the practical implementation. The common weakness of above schemes is, they are not suitable for implementing the gates with larger ξ.

In this paper we investigate a probabilisitic protocol to realize the nonlocal operation

$$U_{AB}(\xi) = e^{i\xi\sigma_{n_A}\sigma_{n_B}}, \xi \in [0, \frac{\pi}{4}] \tag{1}$$

on a general target state $|\Phi_{AB}\rangle$. We show that, by employing the state-operator ("stator") approach [8], a general gate U_{AB} can be implemented with the consumption of less than 1 ebit and 2 classical bits, while the success fidelity maintains a high level. In particular, we prove that gates with $\xi \leq 0.353$ can be implemented with a probability greater than 79.3% and a consumption of less than 0.969 ebits. The fidelity tends to one and the required entanglement tends to zero, as ξ goes to zero. This bound effectively enlarges the region of realizable gates with a more creditable fidelity. The above bound is universal to the residual gates, i.e., any gate with $\xi \geq 0.353$ can be implemented with a fidelity 79.3% and 0.969 ebits, which reaches the same efficiency as the Procrustean method in [14, 16], while our method is more direct and hence the extra bits are saved. Besides, Our method realizes the same effect as that in [14] when ξ is small. Based on the bipartite result above, We generalize some application to the multiple system, where we prove that it is possible to implement certain multiple operations between distributed $non-entangled$ systems by using a non-maximally multiple entangled state. This effect can be realized by adding an intermediator which we will call Charlie. If we measure the required entanglement by dividing the multiple systems into any bipartition, the result will be the same effect as that in bipartite case.

Let us consider two systems A and B at different locations, the collaborators Alice and Bob previously share an entangled state

$$\begin{aligned} |\Psi_{ab_0b_1}\rangle &= \lambda_0 |0_a 0_{b_0} 0_{b_1}\rangle + \lambda_1 |0_a 0_{b_0} 1_{b_1}\rangle \\ &\quad + \lambda_2 |1_a 1_{b_0} 0_{b_1}\rangle + \lambda_3 |1_a 1_{b_0} 1_{b_1}\rangle . \end{aligned} \tag{2}$$

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Here, particle a belongs to Alice and b_0, b_1 belongs to Bob. The co-efficient's $\lambda_i, i = 0, 1, 2, 3$ are non-negative and normalized: $\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2 = 1$. The entanglement is

$$\begin{aligned} E(|\Psi_{ab_0b_1}\rangle) &= -(\lambda_0^2+\lambda_1^2)\log(\lambda_0^2+\lambda_1^2) \\ &\quad -(\lambda_2^2+\lambda_3^2)\log(\lambda_2^2+\lambda_3^2) \\ &\equiv -H\log H - (1-H)\log(1-H), \end{aligned} \quad (3)$$

where $H = \lambda_0^2 + \lambda_1^2$, is the only parameter of entanglement. Here we use $\log(x)$ to denote logarithms to base 2.

First we describe the general technique. Alice and Bob perform the local unitary operation respectively

$$\begin{aligned} U_{aA} &= |0_a\rangle\langle 0_a| \otimes I_A + i\,|1_a\rangle\langle 1_a| \otimes \sigma_{n_A}, \\ U_{b_1B} &= |0_{b_1}\rangle\langle 0_{b_1}| \otimes I_B + |1_{b_1}\rangle\langle 1_{b_1}| \otimes \sigma_{n_B}. \end{aligned} \quad (4)$$

Then Alice measures σ_{xa} and transmits the result to Bob by sending 1 classical bit. Following this message Bob will do nothing or σ_{zb_0}. Thus they get an initial stator

$$\begin{aligned} S_{ini} &= \lambda_0\,|0_{b_0}0_{b_1}\rangle I_A I_B + \lambda_1\,|0_{b_0}1_{b_1}\rangle I_A\sigma_{n_B} \\ &\quad + i\lambda_2\,|1_{b_0}0_{b_1}\rangle \sigma_{n_A} I_B + i\lambda_3\,|1_{b_0}1_{b_1}\rangle \sigma_{n_A}\sigma_{n_B}. \end{aligned} \quad (5)$$

Now Bob collectively measures particle b_0 and b_1 in the following basis

$$\begin{aligned} |B_{00}\rangle &= \cos\delta_0\,|00\rangle + \sin\delta_0\,|11\rangle, \\ |B_{01}\rangle &= \cos\delta_0\,|11\rangle - \sin\delta_0\,|00\rangle, \\ |B_{10}\rangle &= \cos\delta_1\,|01\rangle + \sin\delta_1\,|10\rangle, \\ |B_{11}\rangle &= \cos\delta_1\,|10\rangle - \sin\delta_1\,|01\rangle. \end{aligned} \quad (6)$$

The corresponding probability to get each Basis is

$$\begin{aligned} P(|B_{00}\rangle) &= \lambda_0^2\cos^2\delta_0 + \lambda_3^2\sin^2\delta_0, \\ P(|B_{01}\rangle) &= \lambda_0^2\sin^2\delta_0 + \lambda_3^2\cos^2\delta_0, \\ P(|B_{10}\rangle) &= \lambda_1^2\cos^2\delta_1 + \lambda_2^2\sin^2\delta_1, \\ P(|B_{11}\rangle) &= \lambda_1^2\sin^2\delta_1 + \lambda_2^2\cos^2\delta_1. \end{aligned} \quad (7)$$

After getting one of the resulting operators, Bob broadcasts 1 bit to inform Alice that she will perform σ_{n_A} or nothing. With the local operation σ_{n_B} or I_B by Bob himself, the operators they obtain will be respectively

$$\begin{aligned} S(|B_{00}\rangle) &= \lambda_0\cos\delta_0 I_A I_B + i\lambda_3\sin\delta_0\sigma_{n_A}\sigma_{n_B}, \\ S(|B_{01}\rangle) &= \lambda_3\cos\delta_0 I_A I_B + i\lambda_0\sin\delta_0\sigma_{n_A}\sigma_{n_B}, \\ S(|B_{10}\rangle) &= \lambda_1\cos\delta_1 I_A I_B + i\lambda_2\sin\delta_1\sigma_{n_A}\sigma_{n_B}, \\ S(|B_{11}\rangle) &= \lambda_2\cos\delta_1 I_A I_B + i\lambda_1\sin\delta_1\sigma_{n_A}\sigma_{n_B}. \end{aligned} \quad (8)$$

We show the following results based on the above argument.

(result 1) A simple observation is that, if we set $\lambda_0 = \lambda_3$ and $\lambda_1 = \lambda_2$, while $\delta_0 = \delta_1 = \xi$, which can be decided by Bob. Thus we have faithfully realized the target operation $U_{AB}(\xi) = e^{i\xi\sigma_{n_A}\sigma_{n_B}}$. However we readily get another result, i.e., $H = \frac{1}{2}$, which means that the required entanglement is maximal. Since the classical consumption in this process is 2 bits, we thus reach the same efficiency in [8].

(result 2) On the other hand, if we discard half of the resulting operators above, e.g., we only require $\lambda_0 = \lambda_3$ and $\delta_0 = \xi$. As described above, this fidelity is

$$P(|B_{00}\rangle) + P(|B_{01}\rangle) = 2\lambda_0^2. \quad (9)$$

if we set $\lambda_1 = 0$, thus we could get the same efficiency as that of Procrustean method in [16], where a maximally entangled state will be generated in the intermediate stage with a fidelity, and this EPR singlet is used to implement a general operation [14]. Therefore we can implement the nonlocal gate $U_{AB}(\xi)$ with a fidelity $2\lambda_0^2$ and the entanglement required is $-\lambda_0^2\log\lambda_0^2 - (1-\lambda_0^2)\log(1-\lambda_0^2)$. Besides, our method is more direct than the Procrustean method and thus economizes the extra classical consumption. We call this technique Fast-Procrustean-Transformation (FPT).

(result 3) We adopt another path to explain how to implement gates with nearly unit probability and a small entanglement, when $\xi \to 0$. Here we set $\lambda_0 = \lambda_3 = 0$ and $\lambda_1 = \cos\alpha$, $\lambda_2 = \sin\alpha$. Following the above procedure we will get one of the two operators $S(|B_{10}\rangle)$ and $S(|B_{11}\rangle)$. The probability to get $S(|B_{10}\rangle)$ is

$$P(|B_{10}\rangle) = \cos^2\alpha\cos^2\delta_1 + \sin^2\alpha\sin^2\delta_1. \quad (10)$$

In order to realize $U_{AB}(\xi)$, we set

$$\frac{\sin\alpha\sin\delta_1}{\cos\alpha\cos\delta_1} = \tan\xi. \quad (11)$$

Using this condition we get

$$\begin{aligned} P(|B_{10}\rangle) &= \frac{\sec^2\xi}{\sec^2\alpha + \csc^2\alpha\tan^2\xi} \\ &\le \frac{1}{1+\sin 2\xi}, \end{aligned} \quad (12)$$

where the equality holds when $\tan\alpha = \sqrt{\tan\xi}$. The entanglement resource required for this optimal fidelity is

$$\begin{aligned} E(|\Psi_{ab_0b_1}\rangle) &= -\frac{1}{1+\tan\xi}\log\left(\frac{1}{1+\tan\xi}\right) \\ &\quad -\frac{\tan\xi}{1+\tan\xi}\log\left(\frac{\tan\xi}{1+\tan\xi}\right). \end{aligned} \quad (13)$$

Therefore, we can implement the gate $U_{AB}(\xi)$ with a high fidelity and very small entanglement, when ξ is near zero. This process achieves the same effect as that of [14] and thus it is more efficient than that of [9] for the much saving classical consumption.

In the following, we analyze the implementation of $U_{AB}(\xi) = e^{i\xi\sigma_{n_A}\sigma_{n_B}}$, when ξ is a general amount. We show that, combined with the FPT technique, it is feasible to implement a general nonlocal gate with a high fidelity, by using less than 1 ebit and 2 bit of classical communication.

We still start from the state $|\Psi_{ab_0b_1}\rangle = \lambda_0 |0_a 0_{b_0} 0_{b_1}\rangle + \lambda_1 |0_a 0_{b_0} 1_{b_1}\rangle + \lambda_2 |1_a 1_{b_0} 0_{b_1}\rangle + \lambda_3 |1_a 1_{b_0} 1_{b_1}\rangle$. Here, the coefficient's are defined as

$$\begin{aligned}
\lambda_0 &= \frac{\tan\theta_1 \cos\theta_1}{\sqrt{(\tan^2\theta_0+\tan^2\theta_1)(\cos^2\theta_0+\cos^2\theta_1)}},\\
\lambda_1 &= \frac{\tan\theta_0 \cos\theta_1}{\sqrt{(\tan^2\theta_0+\tan^2\theta_1)(\cos^2\theta_0+\cos^2\theta_1)}},\\
\lambda_2 &= \frac{\tan\theta_0 \cos\theta_0}{\sqrt{(\tan^2\theta_0+\tan^2\theta_1)(\cos^2\theta_0+\cos^2\theta_1)}},\\
\lambda_3 &= \frac{\tan\theta_1 \cos\theta_0}{\sqrt{(\tan^2\theta_0+\tan^2\theta_1)(\cos^2\theta_0+\cos^2\theta_1)}}, \quad (14)
\end{aligned}$$

where the parameters $\theta_0, \theta_1 \in [0, \frac{\pi}{2}]$, and $H = \frac{\cos^2\theta_1}{\cos^2\theta_0+\cos^2\theta_1}$. First Bob implements a POVM operation on particle b_0 as follows

$$M_0 = \begin{pmatrix} \cos\theta_0 & 0 \\ 0 & \cos\theta_1 \end{pmatrix}, M_1 = \begin{pmatrix} \sin\theta_0 & 0 \\ 0 & \sin\theta_1 \end{pmatrix}.$$

We notice that $\lambda_0\cos\theta_0 = \lambda_3\cos\theta_1$,$\lambda_1\cos\theta_0 = \lambda_2\cos\theta_1$. Thus if Bob implements M_0 and we restart from state $M_{0b_0}|\Psi_{ab_0b_1}\rangle$. Following the general technique above, one readily finds this is the case in result (1), i.e., $\lambda_0\cos\theta_0 \to \lambda_0$, $\lambda_3\cos\theta_1 \to \lambda_3$, etc, which means we have faithfully implemented the gate $U_{AB}(\xi)$. On the other hand if Bob implements M_1, then he performs a CNOT gate on particle b_0 and b_1. Thus the resulting state is

$$\begin{aligned}
\left|\Psi^{res}_{ab_0b_1}\right\rangle &= \lambda_0\sin\theta_0 |0_a 0_{b_0} 0_{b_1}\rangle + \lambda_1\sin\theta_0 |0_a 0_{b_0} 1_{b_1}\rangle \\
&\quad + \lambda_3\sin\theta_1 |1_a 1_{b_0} 0_{b_1}\rangle + \lambda_2\sin\theta_1 |1_a 1_{b_0} 1_{b_1}\rangle .
\end{aligned}$$

Again Alice and Bob follow the general technique, starting with the state $\left|\Psi^{res}_{ab_0b_1}\right\rangle$. Notice that $\lambda_0\sin\theta_0 = \lambda_2\sin\theta_1$ and Bob chooses $\delta_0 = \xi$, thus we realize $U_{AB}(\xi)$ on operators $S(|B_{00}\rangle)$ and $S(|B_{01}\rangle)$. Furthermore, we can make operator $S(|B_{10}\rangle) = U_{AB}(\xi)$ by supposing

$$\frac{\lambda_3\sin\theta_1\sin\delta_1}{\lambda_1\sin\theta_0\cos\delta_1} = \tan\xi, \quad (15)$$

or

$$n\tan\xi = \tan\delta_1, n \equiv \frac{\tan^2\theta_0}{\tan^2\theta_1}. \quad (16)$$

Therefore the only failure case is $S(|B_{11}\rangle)$, whose probability is

$$\begin{aligned}
P(|B_{11}\rangle) &= \lambda_1^2\sin^2\theta_0\sin^2\delta_1 + \lambda_3^2\sin^2\theta_1\cos^2\delta_1 \\
&\equiv \frac{1+n^4\tan^2\xi}{(1+n^2\tan^2\xi)(1+n)(1+nb)}. \quad (17)
\end{aligned}$$

Here $b \equiv 2\cot^2\theta_0 + 1$, which should be near one by the fact that

$$H = \frac{\cos^2\theta_1}{\cos^2\theta_0+\cos^2\theta_1} = (\frac{1-n^{-1}}{1+\frac{2}{b-1}} + 1 + n^{-1})^{-1}. \quad (18)$$

That is, the minimum value of $E(|\Psi_{ab_0b_1}\rangle)$ will be attained as b tends to one. In what follows we will keep to this principle in order to make a lowest consumption of entanglement. We take n as the independent variable and calculate the minimum value of $P(|B_{11}\rangle)$ as follows

$$\frac{d}{dn}P(|B_{11}\rangle) = \frac{C_0\tan^4\xi + C_1\tan^2\xi + C_2}{(1+n^2\tan^2\xi)^2(1+n)^2(1+nb)^2},$$

where

$$\begin{aligned}
C_0 &= (2+n+nb)n^5,\\
C_1 &= (2n^3+3n^2-4n-3)n^2b + (3n^3+4n^2-3n-2)n,\\
C_2 &= -1-b-2nb. \quad (19)
\end{aligned}$$

We now explore some conclusions. First, it is easy to check that we have to provide more entanglement than the FPT technique both under an identical success probability when $n \in [0, n_0], n_0 \simeq 1.214$, which satisfies $n^6+2n^5+3n^4-4n^3-3n^2-2n-1=0$. One can compare the two amounts of $E(|\Psi_{ab_0b_1}\rangle)$ with $H_0 = \frac{1-P(|B_{11}\rangle)}{2}$ and $H_1 = (1+n^{-1})^{-1}$ to clarify this result. Second, one can readily checks that $C_2 < 0$ and $C_0 + C_1 + C_2 > 0$ when $n \geq n_0$. Under this condition there exists certain ξ to satisfy $C_0\tan^4\xi + C_1\tan^2\xi + C_2 = 0$, and this is indeed the minimum value of $P(|B_{11}\rangle)$. Again we need to compare this improved technique with the FPT technique under an identical fidelity, which we describe in Figure 1. The parameters therein are as follows (We set $b = 1.001$ in this figure and in fact this parameter tends to one. Thus we take the following expression of E_0, which approximates the image in figure 1).

$$\begin{aligned}
E_{FPT} &= -H_0\log H_0 - (1-H_0)\log(1-H_0),\\
E_0 &= -H_1\log H_1 - (1-H_1)\log(1-H_1),\\
\tan^2\xi &= -\frac{C_1}{2C_0} + \sqrt{\frac{C_1^2}{4C_0^2} - \frac{C_2}{C_0}},\\
F &= 1 - P(|B_{11}\rangle). \quad (20)
\end{aligned}$$

From the comparison we see that it is more efficient to employ the improved technique to implement the gate $U_{AB}(\xi)$ when $\xi \leq 0.353$. On the other hand, one still employs the FPT technique when $\xi > 0.353$. Since the FPT technique is universal to any ξ, we thus get the conclusion that any gate $U_{AB}(\xi)$ with $\xi > 0.353$ can be implemented by the consumption of 0.969 ebit and 2 cbits, under a fidelity 79.3%. One will see the success fidelity we obtain is much higher than that in [14], while our entanglement consumption is also lower than one ebit and tends to zero as ξ goes to zero. Furthermore, compared to [9] our method saves entanglement when ξ increases, e.g., when $\xi = 0.17$ the required ebit here is 0.897 with a fidelity 85.6%, while it will be 1.016 ebits in [9].

In the following we generalize some useful application to the multiple gate. First we review the general technique, where we employ the entangled state $|\Psi_{ab_0b_1}\rangle = \lambda_0 |0_a 0_{b_0} 0_{b_1}\rangle + \lambda_1 |0_a 0_{b_0} 1_{b_1}\rangle + \lambda_2 |1_a 1_{b_0} 0_{b_1}\rangle + \lambda_3 |1_a 1_{b_0} 1_{b_1}\rangle$.

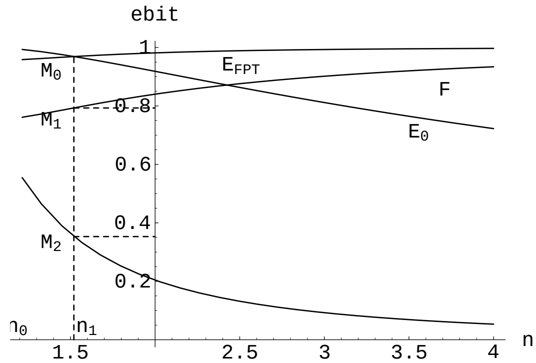

FIG. 1: Comparison of entanglement consumption E_0 and E_{FPT} between the improved technique and the FPT technique, under a common success fidelity $F = 1 - P(|B_{11}\rangle)$ and $b = 1.001$. The lowest curve represents ξ, which increases as n tends to n_0. $E_{FPT} = E_0 \simeq 0.969$ at the point M_0, where $n_1 \simeq 1.521$. The lowest fidelity $F = 0.793$, is obtained at the point M_1. The corresponding value of ξ approximates 0.353 at the point M_2.

Now we introduce an assistant called Charlie and the three partners share such entangled state instead

$$\begin{aligned} |\Psi_{abc_0c_1}\rangle &= \lambda_0 |0_a 0_b 0_{c_0} 0_{c_1}\rangle + \lambda_1 |0_a 0_b 0_{c_0} 1_{c_1}\rangle \\ &\quad + \lambda_2 |1_a 1_b 1_{c_0} 0_{c_1}\rangle + \lambda_3 |1_a 1_b 1_{c_0} 1_{c_1}\rangle, \end{aligned} \tag{21}$$

where particle a belongs to Alice, b belongs to Bob and Charlie possesses c_0 and c_1. The difference between $|\Psi_{abc_0c_1}\rangle$ and $|\Psi_{ab_0b_1}\rangle$ is: in state $|\Psi_{abc_0c_1}\rangle$, the assistant Charlie replaces the role of Bob in state $|\Psi_{ab_0b_1}\rangle$. Therefore we can carry out a scheme similar to that of general technique by using $|\Psi_{abc_0c_1}\rangle$ like this. First Alice and Bob locally perform U_{aA} and U_{bB} on their particles respectively, followed by the measurement of σ_{xa} and σ_{xb}. Now they get another initial stator

$$\begin{aligned} S'_{ini} &= \lambda_0 |0_{c_0} 0_{c_1}\rangle I_A I_B + \lambda_1 |0_{c_0} 1_{c_1}\rangle I_A I_B \\ &\quad + i\lambda_2 |1_{c_0} 0_{c_1}\rangle \sigma_{n_A}\sigma_{n_B} + i\lambda_3 |1_{c_0} 1_{c_1}\rangle \sigma_{n_A}\sigma_{n_B}. \end{aligned} \tag{22}$$

The remainder of this process is the collective measurement on particle c_0 and c_1 by Charlie and all analysis above works. In the case of the improved technique, the POVM and CNOT gate therein will be implemented by Charlie instead. This completes the whole argument using state $|\Psi_{abc_0c_1}\rangle$. Therefore it is also feasible to employ the state $|\Psi_{abc_0c_1}\rangle$ to implement the nonlocal gate $U_{AB}(\xi) = e^{i\xi\sigma_{n_A}\sigma_{n_B}}, \xi \in [0, \frac{\pi}{4}]$, while the classical consumption is 4 bits here. However one will notice that, there is no entanglement between any two partners in the state $|\Psi_{abc_0c_1}\rangle$, in fact any two of them are separable [17]. This is similar to the GHZ qutrit and thus we call $|\Psi_{abc_0c_1}\rangle$ the $quasi-GHZ$ state. As we know, the GHZ state is the maximally entangled state in multiple system. Here we regard the $quasi-GHZ$ state as the $non-maximally$ entangled state in multiple system by diving the multiple system into a random bipartite system, namely A-BC, B-AC or C-AB. One readily gets the entropy of any system above will be $-H\log H - (1-H)\log(1-H)$, which is less than one except $H = \frac{1}{2}$. Since $H \neq \frac{1}{2}$ in the bipartite results, we also employ the non-maximally entangled state in the multiple case [18].

The above argument implies that one can implement the nonlocal gate on a pair of partners who are not entangled by adding a third company Charlie, who indeed works as an *intermediator*. The major virtue of this mode is, Alice and Bob will only need to know the content of σ_{nA} and $\sigma_{n'B}$ respectively(we notice it is not necessarily that the two directions n's are always the same). Therefore, the knowledge of ξ can be unclear to both Alice and Bob, while Bob must hold all knowledge of the state $|\Psi_{ab_0b_1}\rangle$ and ξ, σ_{nB} in the general technique above, which is also a universal status in existed work. Second, if Charlie performs a local operation

$$U_{c_0C} = |0_{c_0}\rangle\langle 0_{c_0}| \otimes I_C + |1_{c_0}\rangle\langle 1_{c_0}| \otimes \sigma_{n_C} \tag{23}$$

on the stator S'_{ini} where C is the target particle of Charlie, this will lead to the realization of gate $U_{ABC}(\xi) = e^{i\xi\sigma_{n_A}\sigma_{n_B}\sigma_{n_C}}, \xi \in [0, \frac{\pi}{4}]$. One can also generalize the $quasi-GHZ$ state to a multiple case, that is

$$\begin{aligned} |\Psi_{a_1a_2\ldots a_N c_0 c_1}\rangle &= \lambda_0 |0_{a_1} 0_{a_2} \ldots 0_{a_N} 0_{c_0} 0_{c_1}\rangle \\ &\quad + \lambda_1 |0_{a_1} 0_{a_2} \ldots 0_{a_N} 0_{c_0} 1_{c_1}\rangle \\ &\quad + \lambda_2 |1_{a_1} 1_{a_2} \ldots 1_{a_N} 1_{c_0} 0_{c_1}\rangle \\ &\quad + \lambda_3 |1_{a_1} 1_{a_2} \ldots 1_{a_N} 1_{c_0} 1_{c_1}\rangle. \end{aligned} \tag{24}$$

Here N particles $a_i, i = 1, 2, ..., N$ are spatially distributed in N partners respectively, which we call $A_i, i = 1, 2, ..., N$. Following the same technique above, we are able to realize the gate $U_{A_1A_2\ldots A_NC}(\xi) = \exp[i\xi\sigma_{n_{A_1}}\sigma_{n_{A_2}}\ldots\sigma_{n_{A_N}}\sigma_{n_C}], \xi \in [0, \frac{\pi}{4}]$, where the classical consumption will be $2N$ bits which equals to the amount in [8]. Again, each participant only needs to know the corresponding gate σ_n, except Charlie. Besides, comparing with the technique in [8] our scheme reduces the entanglement consumption for implementing the gate $U_{A_1A_2\ldots A_NC}(\xi)$, if we divide the multiple system into random bipartite system. In particular, the entanglement between Charlie and the N distributed receivers remains less than 1 ebit in our scheme, while it requires N ebits in [8].

In conclusion, we have given an explicit scheme to realize the nonlocal operation $U_{AB}(\xi) = e^{i\xi\sigma_{n_A}\sigma_{n_B}}, \xi \in [0, \frac{\pi}{4}]$ by using a single non-maximally entangled state. The technique introduced here allows one to implement the nonlocal gate with the consumption of less than 0.969 ebits and 2 cbits, and the minimum success fidelity is 79.3%. The entanglement consumption will decrease to zero and the fidelity goes to one as ξ ranges from 0.353 to zero, which enlarges the region of applicable gates and effectively saves entanglement and operates more reliably compared to the former techniques. The classical communication is also the lowest amount so far. An open question is, whether there is more efficient technique than

the FPT way in the region $\xi \geq 0.353$. The required entanglement here can be further reduced if lower fidelity is allowed. We also provide another method to perfectly implement the gate with a small ξ, which reaches the same effect in [14]. Besides, we generalize this technique to the implementation of multiple gate also using non-maximally multiple entangled state and in a sense it largely reduces the required entanglement in the multiple case. The fact that nonlocal gate could be performed between non-entangled partners is also notable, which could be useful to other tasks such as quantum secret sharing.

The work was partly supported by the NNSF of China (Grant No.90203003), NSF of Zhejiang Province (Grant No.602018), and by the Foundation of Education Ministry of China (Grant No.010335025).

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[18] It is noticeable that, so far there is no specific definition of non-maximally entangled state in multiple system. In this paper the employed entangled state has a general form $|\Psi_{a_1a_2...a_Nc_0c_1}\rangle$ described in the text, which could be made out from a quNit $\sqrt{\lambda_0^2+\lambda_1^2}\,|0_{a_1}0_{a_2}...0_{a_N}0_{c_0}0_{c_1}\rangle + \sqrt{\lambda_2^2+\lambda_3^2}\,|1_{a_1}1_{a_2}...1_{a_N}1_{c_0}1_{c_1}\rangle$, by locally collective unitary operation by Charlie. This process preserves the entanglement. It is easy to see that this quNit is equivalent to the standard N-GHZ state under SLOCC [19, 20]. Therefore it is reasonable to compare the entanglement between the above quNit and the N-GHZ state, by dividing the multiple system into any bipartite system including the same parties. As described in the text(the example of N=2), this quNit is less entangled than the GHZ state and it is non-maximally.
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